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Energy states of colored particle in a chromomagnetic field

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Abstract. The unitary transformation which diagonalizes the squared Dirac equation in a constant chromomagnetic field is found. Applying this transformation, we find the eigenfunctions of the diagonalized Hamiltonian, that describes the states with a definite value of energy, and we call them energy states. It is pointed out that the energy states are determined by the color interaction term of the particle with the background chromofield, and this term is responsible for the splitting of the energy spectrum. We construct supercharge operators for the diagonal Hamiltonian that ensure the superpartner property of the energy states.

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1 Introduction

Classical color field configurations are important for the study of theoretical problems of non-Abelian charged particles. These problems are connected with QCD problems [1–6, 17–36], and their study gives valuable information on different effects in QCD.

Because color magnetic fields have special significance for the vacuum state of QCD [1–4], in [7] and [8] one considered the motion of colored particle in such fields. For giving a constant and homogeneous color background the constant vector potentials introduced in [9, 10] were applied, and the squared Dirac equation was solved for this problem. The similarity of this motion to the motion of an electron in an ordinary magnetic field is that in both cases we have circular orbits for the motion in the uniform (chromo-) magnetic field and well-known s, p, d, f, ... orbitals in a background field having spherical symmetry. But the energy spectrum in the non-Abelian problem differs from the one in the Abelian case, and in the quantized spectrum case it does not look like Landau levels. This is entirely connected with the matrix structure of the color interaction and the color matrix structure of the quantum mechanics of non-Abelian charged particles.

In addition to non-diagonal spin matrices the Dirac equation for these particles contains non-diagonal color matrices. These matrices mix different color and spin components of the wave function in the equations for these components, which are obtained from the squared Dirac equation. As a result of such mixing, we cannot write an eigenvalue equation separately for each color state and for each spin state in the general case of the background field. Therefore, we are not able to have the known wave functions correspond to the spectrum and thus to determine the energy of each state with a definite value of spin and color spin. We neither have a wave function for the description of the states with definite energy. The way out of this situation is to diagonalize the squared Dirac operator in the color spin or combined spin–color spin spaces. For this we should find a unitary transformation diagonalizing this operator. The wave function of the particle, as a matrix in these spaces, also will transform under this transformation. New transformed states will obey the eigenvalue equation for the diagonalized squared Dirac operator. We aim to find this transformation and by means of it to construct the states – eigenfunctions of the diagonalized squared Dirac equation for the considered case of a constant chromomagnetic background.

Another property of problems in quantum mechanics – supersymmetry in the Dirac equation – has been considered before for this kind of chromomagnetic background field as well [18, 19, 50]. It is reasonable to reconsider the supersymmetry for the diagonalized Hamiltonian and to construct the supercharge operators for the diagonal representation. Knowing the states with definite energy gives us the possibility to study the question of superpartner states in this supersymmetry and thus to treat supersymmetry as the origin of the spectrum degeneration. We suppose that the new supercharge operators will ensure the superpartnership of the energy states.

2 Axial chromomagnetic field

The Dirac equation for a colored particle in an external color field is obtained from the one for a free particle by a

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momentum shift:

$$(\gamma^\mu P_\mu - M)\psi = 0\,, \tag{1}$$

where $P_\mu = p_\mu + gA_\mu = p_\mu + gA^a_\mu \lambda^a/2$; the λ^a are the Gell–Mann matrices describing the particle's color spin, and within the $SU_c(3)$ color symmetry group g is the color interaction constant and the color index a runs over $a = 1, \ldots, 8$. Written for the Majorana spinors ϕ and χ (1) we have the well-known form

$$\left(\sigma^i P_i\right)^2 \psi = -\left(\frac{\partial^2}{\partial t^2} + M^2\right)\psi\,, \tag{2}$$

where the Pauli matrices σ^i describe the particle's spin. Hereafter ψ means ϕ or χ. The spinors ϕ and χ have two components corresponding to the two spin states of a particle, $\psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$. Each component of ψ transforms under the fundamental representation of the color group $SU_c(3)$ and has three color components describing the color states of a particle corresponding to the three eigenvalues of the color spin λ^3:

$$\psi_\pm = \begin{pmatrix}\psi_\pm(\lambda^3 = +1)\\ \psi_\pm(\lambda^3 = -1)\\ \psi_\pm(\lambda^3 = 0)\end{pmatrix} = \begin{pmatrix}\psi^{(1)}_\pm\\ \psi^{(2)}_\pm\\ \psi^{(3)}_\pm\end{pmatrix}. \tag{3}$$

We are going to continue the study of the motion in the chromomagnetic field started in [7], where we applied the constant vector potentials introduced in [9, 10]. Recall that for giving an axial chromomagnetic field by a constant vector potential, the components of the latter are chosen as follows:

$$A^a_1 = \sqrt{\tau}\delta_{1a}\,, \quad A^a_2 = \sqrt{\tau}\delta_{2a}\,, \quad A^a_3 = 0\,, \quad A^a_0 = 0\,, \tag{4}$$

where τ is a constant and $\delta_{\mu a}$ is the Kronecker symbol.

For this A^a_μ all components of the field strength tensor $F^c_{\mu\nu} = gf^{abc}A^a_\mu A^b_\nu$ are equal to zero, except for

$$F^3_{12} = g\tau = H^3_z\,, \tag{5}$$

and (4) gives a constant chromomagnetic field directed along the third axes of the ordinary and color spaces. Here the f^{abc} are the structure constants of the $SU_c(3)$ group.

Setting (4) and $\mathrm{i}\partial\psi/\partial t = E\psi$ in (2), we obtain the following two equations for the $\psi_\pm$, which differ from each other by the sign of the last term in the left hand side:

$$\left[\mathbf{p}^2 + \frac{1}{2}g^2\tau I_2 + g\tau^{1/2}\left(p_1\lambda^1 + p_2\lambda^2 \mp \frac{1}{2}g\tau^{1/2}\lambda^3\right)\right]\psi_\pm = (E^2 - M^2)\psi_\pm\,. \tag{6}$$

Here $I_2 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ is the color matrix. Because the field (5) is directed along the third axis, $\psi_\pm$ describes the spin states with the up and down projections of σ^3 and the Hamiltonians defined as $H_\pm\psi_\pm = E^2\psi_\pm$ correspond to these spin states. Because the non-diagonal λ^1 and λ^2 matrices $H_\pm$ have no diagonal color structure, we cannot write the eigenvalue equation

$$H_\pm\psi^{(i)}_\pm = E^2\psi^{(i)}_\pm$$

for pure color states $\psi^{(i)}_\pm$, except for the states $\psi^{(3)}_\pm$. The explicit matrix form of the general Hamiltonian in the combined color and spin spaces is

$$H = \begin{pmatrix}
\mathcal{P}^2 & \mathcal{G}p_- & 0 & 0 & 0 & 0\\
\mathcal{G}p_+ & \mathcal{P}^2+\mathcal{G}^2 & 0 & 0 & 0 & 0\\
0 & 0 & \mathcal{P}^2 & 0 & 0 & 0\\
0 & 0 & 0 & \mathcal{P}^2+\mathcal{G}^2 & \mathcal{G}p_- & 0\\
0 & 0 & 0 & \mathcal{G}p_+ & \mathcal{P}^2 & 0\\
0 & 0 & 0 & 0 & 0 & \mathcal{P}^2
\end{pmatrix}. \tag{7}$$

For brevity we have introduced the notation $\mathcal{P}^2 = \mathbf{p}^2 + M^2$, $\mathcal{G} = g\tau^{1/2}$, $p_\pm = p_1 \pm \mathrm{i}p_2$. Non-diagonality of (7) leads to mixing of the different color states $\psi^{(i)}$ in the differential equations obtained from (6). But the equation for the pure states $\psi^{(1),(2)}_\pm$ obtained from (7) has the same form for all the color and spin states [7]. In the cylindrical coordinates the common equation for $\psi^{(1),(2)}_\pm$ has a solution expressed by the Bessel function $J_m(x)$:

$$\psi^{(i)}_\pm(\mathbf{r}) = \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}m\varphi}\exp(\mathrm{i}p_3 z)J_m(p_\perp r)\xi^{(i)}_\pm\,. \tag{8}$$

Here m is the chromomagnetic quantum number and $\xi^{(i)}_\pm$ includes the spin and color spin parts of the wave function. Similarly to the spin part of the wave function, $\xi^{(i)}_\pm$ can be chosen as follows:

$$\xi^{(i)}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\zeta^{(i)}, \quad \xi^{(i)}_- = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\zeta^{(i)}\,;$$

$$\zeta^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \zeta^{(2)} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \zeta^{(3)} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Thus chosen, ξ obeys the normalizing condition

$$\left|\xi^{(i)}_+\right|^2 + \left|\xi^{(i)}_-\right|^2 = |\zeta^{(1)}|^2 + |\zeta^{(2)}|^2 + |\zeta^{(3)}|^2 = 1\,.$$

Let us remind the reader that in spite of the same r-dependence of all these states, the different $\psi^{(i)}_\pm(\mathbf{r})$ transforms differently under transformations in the spin and color spaces. This solution is similar to the one for relativistic motion of an electron in an axial magnetic field [11] and in the classical picture gives motion on circular orbits [7]. The energy spectrum for (6) was found by solving the determinant equation obtained from it and has three continuous branches in the case of infinite motion [14–17]:

$$E^2_{1,2} = (\mathcal{P}_\perp \mp \mathcal{G}/2)^2 + p_3^2 + M^2, \quad E^2_3 = \mathcal{P}^2, \tag{9}$$

where $\mathcal{P}_\perp = \sqrt{p_\perp^2 + \mathcal{G}^2/4}$, $p_\perp^2 = p_1^2 + p_2^2$. The equation for the state $\psi^{(3)}_\pm(\mathbf{r})$ is the equation for the free particle

$\mathcal{P}^2\psi_\pm^{(3)}(\mathbf{r}) = (\mathbf{p}^2 + M^2)\psi_\pm^{(3)}(\mathbf{r})$ and the solution (8) can be regarded as well.

For the motion limited by a cylinder with radius r_0, the quantized spectrum has the form [7]

$$\left(E_m^{(N)}\right)^2_{1,2} = \left(\sqrt{\left(\frac{\alpha_m^{(N)}}{r_0}\right)^2 + \frac{\mathcal{G}^2}{4}} \mp \mathcal{G}/2\right)^2 + p_3^2 + M^2,$$
$$\left(E_m^{(N)}\right)^2_{3} = \left(\frac{\alpha_m^{(N)}}{r_0}\right)^2 + p_3^2 + M^2.$$

Here $\alpha_m^{(N)}$ are the Bessel function's zeros, and N is the radial quantum number. The finite motion solution is connected with the one for the infinite motion by the replacement $p_\perp = \alpha_m^{(N)}/r_0$ in (8) and for the wave function we do not sum over the m inasmuch as quantized energy levels are determined by this quantum number:

$$\psi_\pm^{(i)}(\mathbf{r}) = \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}m\varphi}\sin(p_3 z)J_m\left(\alpha_m^{(N)}r/r_0\right)\xi_\pm^{(i)}.$$

Hereafter, by $\mathbf{p}$ and the spectrum E_i^2 we shall mean both their continuous and quantized values. The spectrum found in such a way is not determined by the value of the projection of the color spin operator λ^3 onto the field (5). This means that the states $\psi_\pm^{(1),(2)}$ have not definitely gotten the energy from the branches E_1 or E_2. They can get energy from both branches of the energy spectrum (9), but with a different probability. Then the question arises as to what the wave functions are of states having a definite energy from the branch E_1 or from the branch E_2. It is clear that for such states we would be able to write the eigenvalue equation with the spectrum branches (9), and this equation can be written only for the diagonal matrix form of the Hamiltonian. So we need the diagonal form of the Hamiltonian (7) in the combined spin–color spin space, in order to write eigenvalue equation by it and then to determine its eigenfunctions. Since the Hamiltonian is a hermitian matrix, it has a diagonal form in the basis of its eigenfunctions, and this diagonal form is unique. We can find the diagonal form of H' for the Hamiltonian (7) and then its diagonal elements will correspond to the spectrum branches from (9). By H' we will be able to write an eigenvalue equation with the eigenvalues from (9):

$$H'\Psi' = E_k^2\Psi'. \tag{10}$$

Here Ψ' is the eigenfunction of H', which is different from $\psi_\pm^{(1),(2)}$. In order to get the diagonal H' from the nondiagonal Hamiltonian H (7), we should make some transformation U in the combined spin–color spin space. The wave functions $\psi_\pm^{(i)}$ will be transformed by this transformation as well. More precisely, the Hamiltonian (7) will get the diagonal form under some U transformation of the basis vectors of the combined spin–color spin space. The basis vectors in the combined space are the solutions $\psi_\pm^{(i)}$ (their $\xi_\pm^{(i)}$ part), since the eigenfunctions of the spin and color spin operators σ^i and λ^a are these functions. Under U transformation of the combined space, the basis vectors, i.e. $\psi_\pm^{(i)}$, transform according to the rule

$$\Psi' = U\Psi\,, \quad \Psi = \begin{pmatrix} \psi_+^{(1)} \\ \psi_+^{(2)} \\ \psi_+^{(3)} \\ \psi_-^{(1)} \\ \psi_-^{(2)} \\ \psi_-^{(3)} \end{pmatrix}. \tag{11}$$

The components of the new basis vector Ψ' will not be states with definite values of the projection of spin or color spin and will be some superposition of the $\psi_\pm^{(i)}$. As we stated above, if H' is diagonal, then the basis vectors of this space are the eigenvectors of this Hamiltonian. So these components are the states with definite values of the energy, because they are eigenfunctions of H' with the eigenvalue E_k^2 from (9). We call the components of Ψ' the *energy states* because of describing the states having a definite energy from the branches E_k. Since H' is unique, the transformation matrix U and the basis vector Ψ' are unique as well.[1]

Under transformation (11) the Hamiltonian (7), as any matrix in this space, transforms by a similarity transformation:

$$U^{-1}HU = H'. \tag{12}$$

The difficulty of determining H' is that we have no explicit form of either U or H'; two of three matrices in (12) are unknown. But, fortunately, it turns out to be possible to find both of these matrices, relying on their properties. Firstly, from the hermiticity of H' we can conclude that the transformation matrix U, in addition to being unique, should be unitary:

$$H'^{\dagger} = U^{\dagger}H^{\dagger}U^{-1\dagger} = H' = U^{-1}HU\,,$$
$$U^{-1\dagger} = U \Rightarrow UU^{\dagger} = 1\,.$$

Equation (12) is the basic relation between two Hamiltonian matrices H and H'. Multiplied from the right hand side by U it has got a more useful form for solving, since it contains only the linear relations between the elements u_{ij} of the U matrix:

$$HU = UH'. \tag{13}$$

Another important point for finding H' and U is that the unitary transformation does not change the determinant and trace of a matrix:

$$\det H' = \det U^{-1}\det H\det U = \det H\,, \quad \mathrm{Tr}H = \mathrm{Tr}H'. \tag{14}$$

The matrices H, H' and U are 6×6 matrices. Due to the quasidiagonal form of (7) we can separately consider two

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[1] We have ignored that the wave functions in quantum mechanics are defined up to phase factor. We shall come to this circumstance later.

3×3 Hamiltonians $H_\pm$, instead of a 6×6 one. Of course, (13) and (14) will hold for both of them, but with different matrices $U_\pm$ respectively. The matrices $U_\pm$ belong to the color symmetry group $SU_c(3)$, so they obey the unimodularity condition as well. Solving equations obtained from (13) jointly with the equations of unitarity and unimodularity we find[2] the $U_\pm$ matrices explicitly:

$$U_\pm = \left(\frac{1}{2\mathcal{P}_\perp}\right)^{1/2} \times \begin{pmatrix} (\mathcal{P}_\perp+\mathcal{G}/2)^{1/2}\mathrm{e}^{\mathrm{i}\alpha} & \pm p_-(\mathcal{P}_\perp+\mathcal{G}/2)^{-1/2}\mathrm{e}^{-\mathrm{i}\alpha} & 0 \\ \mp p_+(\mathcal{P}_\perp+\mathcal{G}/2)^{-1/2}\mathrm{e}^{\mathrm{i}\alpha} & (\mathcal{P}_\perp+\mathcal{G}/2)^{1/2}\mathrm{e}^{-\mathrm{i}\alpha} & 0 \\ 0 & 0 & (2\mathcal{P}_\perp)^{1/2} \end{pmatrix} \tag{15}$$

and establish the diagonal elements of $H'_\pm$, which respectively are equal to

$$h'_{11} = \mathcal{P}^2 - \mathcal{G}\mathcal{P}_\perp + \mathcal{G}^2/2\,, \quad h'_{22} = \mathcal{P}^2 + \mathcal{G}\mathcal{P}_\perp + \mathcal{G}^2/2\,, \quad h'_{33} = \mathcal{P}^2 \tag{16}$$

and

$$h''_{11} = \mathcal{P}^2 + \mathcal{G}\mathcal{P}_\perp + \mathcal{G}^2/2\,, \quad h''_{22} = \mathcal{P}^2 - \mathcal{G}\mathcal{P}_\perp + \mathcal{G}^2/2\,, \quad h''_{33} = \mathcal{P}^2. \tag{17}$$

According to (12) the matrix

$$\mathcal{U} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}$$

will reduce the Hamiltonian (7) to the following diagonal form:

$$H' = \begin{pmatrix} h'_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & h'_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h'_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & h''_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & h''_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & h''_{33} \end{pmatrix}. \tag{18}$$

In spite of the $\mathcal{U}$ matrix being a transformation of 6-dimensional spin–color spin space, indeed it transforms the color space, because it has a quasidiagonal form and does not mix spin indices. Comparing the explicit forms of the diagonal elements (16) and (17) with the energy branches (9) the correspondence between them can easily be established:

$$\begin{aligned} h'_{22}\,,\, h''_{11} &\longrightarrow E_1^2\,, \\ h'_{11}\,,\, h''_{22} &\longrightarrow E_2^2\,, \\ h'_{33}\,,\, h''_{33} &\longrightarrow E_3^2\,. \end{aligned} \tag{19}$$

According to (11) we construct the wave function Ψ', which will be an eigenfunction of the Hamiltonian (18):

$$\Psi' = \mathcal{U} \begin{pmatrix} \psi_+^{(1)} \\ \psi_+^{(2)} \\ \psi_+^{(3)} \\ \psi_-^{(1)} \\ \psi_-^{(2)} \\ \psi_-^{(3)} \end{pmatrix} = \begin{pmatrix} \psi_+^{(+)} \\ \psi_+^{(-)} \\ \psi_+^{(3)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_-^{(3)} \end{pmatrix}. \tag{20}$$

Due to the correspondence (19) the desired eigenvalue equation for H' and Ψ' (10) can be written in the following explicit form:

$$\begin{pmatrix} h'_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & h'_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h'_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & h''_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & h''_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & h''_{33} \end{pmatrix} \begin{pmatrix} \psi_+^{(+)} \\ \psi_+^{(-)} \\ \psi_+^{(3)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_-^{(3)} \end{pmatrix} = \begin{pmatrix} E_2^2\psi_+^{(+)} \\ E_1^2\psi_+^{(-)} \\ E_3^2\psi_+^{(3)} \\ E_1^2\psi_-^{(+)} \\ E_2^2\psi_-^{(-)} \\ E_3^2\psi_-^{(3)} \end{pmatrix}.$$

The correspondence between the energy branches E_k and the components $\psi_\pm^{(\pm)}$, which we have called the energy states, can be found from the right hand side of the last equality:

$$\begin{aligned} E_1^2 &\longrightarrow \psi_+^{(-)}, \psi_-^{(+)}\,, \\ E_2^2 &\longrightarrow \psi_+^{(+)}, \psi_-^{(-)}\,, \\ E_3^2 &\longrightarrow \psi_+^{(3)}, \psi_-^{(3)}\,. \end{aligned} \tag{21}$$

Having set in (20) the explicit form of the matrix $\mathcal{U}$ we find the wave functions of the states having energy E_k as a superposition of spin–color spin states $\psi_\pm^{(a)}$ of the non-transformed space:

$$\begin{aligned} \psi_\pm^{(+)} &= u_{11}\psi_\pm^{(1)} + u_{12}\psi_\pm^{(2)} \\ &= \left(2\widehat{\mathcal{P}}_\perp\right)^{-1/2}\left(\widehat{\mathcal{P}}_\perp + \mathcal{G}/2\right)^{1/2}\mathrm{e}^{\mathrm{i}\alpha}\psi_\pm^{(1)} \\ &\quad \pm\hat{p}_-\left(2\widehat{\mathcal{P}}_\perp\right)^{-1/2}\left(\widehat{\mathcal{P}}_\perp + \mathcal{G}/2\right)^{-1/2}\mathrm{e}^{-\mathrm{i}\alpha}\psi_\pm^{(2)}\,, \\ \psi_+^{(-)} = \psi_-^{(-)} &= u_{21}\psi_\pm^{(1)} + u_{22}\psi_\pm^{(2)} \\ &= \mp\hat{p}_+\left(2\widehat{\mathcal{P}}_\perp\right)^{-1/2}\left(\widehat{\mathcal{P}}_\perp + \mathcal{G}/2\right)^{-1/2}\mathrm{e}^{\mathrm{i}\alpha}\psi_\pm^{(1)} \\ &\quad + \left(2\widehat{\mathcal{P}}_\perp\right)^{-1/2}\left(\widehat{\mathcal{P}}_\perp + \mathcal{G}/2\right)^{1/2}\mathrm{e}^{-\mathrm{i}\alpha}\psi_\pm^{(2)}\,. \end{aligned}$$

Having written the momentum operators $\widehat{p}_i$ in the polar coordinates r and φ

$$\hat{p}_1 \pm \mathrm{i}\hat{p}_2 = -\mathrm{i}\mathrm{e}^{\pm\mathrm{i}\varphi}\left(\frac{\partial}{\partial r} \pm \frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi}\right),$$

and applying the recurrent formula for differentiation of the Bessel functions $J_m(x)$,

$$J'_m(x) = \frac{m}{x}J_m(x) - J_{m+1}(x) = J_{m-1}(x) - \frac{m}{x}J_m(x),$$

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[2] Details of this finding are described in [52] or in the next section for the case considered there.

we easily establish the action of these operators on the Bessel functions:

$$\begin{aligned}
\big(\hat p_1 \pm \mathrm{i}\hat p_2\big) J_m(p_\perp r)\mathrm{e}^{\mathrm{i}m\varphi} &= \pm \mathrm{i}p_\perp J_{m\pm 1}(p_\perp r)\mathrm{e}^{\mathrm{i}(m\pm 1)\varphi},\\
\hat p_\perp^2 J_m(p_\perp r)\mathrm{e}^{\mathrm{i}m\varphi} &= \big(\hat p_1 \pm \mathrm{i}\hat p_2\big)\big(\hat p_1 \mp \mathrm{i}\hat p_2\big) J_m(p_\perp r)\mathrm{e}^{\mathrm{i}m\varphi}\\
&= p_\perp^2 J_m(p_\perp r)\mathrm{e}^{\mathrm{i}m\varphi},\\
\widehat{\mathcal{P}}_\perp \psi_\pm^{(i)} &= \sqrt{\hat p_\perp^2 + g^2\tau/4}\,\psi_\pm^{(i)} = \mathcal{P}_\perp \psi_\pm^{(i)}\,.
\end{aligned} \tag{22}$$

As is seen the action of the operators $\hat p_1 \pm \mathrm{i}\hat p_2$ shift the state with m to the state with $m \pm 1$. This means that, in the quantized spectrum case, when the transverse momentum $p_\perp$ gets values determined by the chromomagnetic quantum number m, $p_\perp = \alpha_m^{(N)}/r_0$, and we do not sum over this quantum number, the energy states will be a superposition of the states with the different values of m:

$$\begin{aligned}
\psi_\pm^{(+)} &= \frac{\sin(p_3 z)}{\sqrt{2\mathcal{P}_\perp}}\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\big[(\mathcal{P}_\perp + \mathcal{G}/2)^{1/2}\mathrm{e}^{\mathrm{i}\alpha} J_m(p_\perp r)\zeta_\pm^{(1)}\\
&\quad \mp \mathrm{i}p_\perp(\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2}\mathrm{e}^{-\mathrm{i}\alpha}J_{m-1}(p_\perp r)\mathrm{e}^{-\mathrm{i}\varphi}\zeta_\pm^{(2)}\big]\,,\\
\psi_\pm^{(-)} &= \frac{\sin(p_3 z)}{\sqrt{2\mathcal{P}_\perp}}\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\\
&\quad\times\big[\mp \mathrm{i}p_\perp(\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2}\mathrm{e}^{\mathrm{i}\alpha}J_{m+1}(p_\perp r)\mathrm{e}^{\mathrm{i}\varphi}\zeta_\pm^{(1)}\\
&\quad + (\mathcal{P}_\perp + \mathcal{G}/2)^{1/2}\mathrm{e}^{-\mathrm{i}\alpha}J_m(p_\perp r)\zeta_\pm^{(2)}\big]\,.
\end{aligned}$$

In the case of a continuous spectrum, the transverse momentum $p_\perp$ and the spectrum do not depend on m and we can sum over this quantum number in the energy states, as we did in (8). This just leads to a replacement of the momentum operators by their eigenvalues, and then the energy states have got the form

$$\begin{aligned}
\psi_\pm^{(+)} &= \frac{1}{\sqrt{2\mathcal{P}_\perp}}(\mathcal{P}_\perp + \mathcal{G}/2)^{1/2}\mathrm{e}^{\mathrm{i}\alpha}\psi_\pm^{(1)}\\
&\quad \mp \frac{\mathrm{i}p_\perp}{\sqrt{2\mathcal{P}_\perp}}(\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2}\mathrm{e}^{-\mathrm{i}\alpha}\psi_\pm^{(2)},\\
\psi_\pm^{(-)} &= \mp \frac{\mathrm{i}p_\perp}{\sqrt{2\mathcal{P}_\perp}}(\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2}\mathrm{e}^{\mathrm{i}\alpha}\psi_\pm^{(1)}\\
&\quad + \frac{1}{\sqrt{2\mathcal{P}_\perp}}(\mathcal{P}_\perp + \mathcal{G}/2)^{1/2}\mathrm{e}^{-\mathrm{i}\alpha}\psi_\pm^{(2)}.
\end{aligned} \tag{23}$$

It can be found from (23) that all energy states have wave functions with the same modulus:

$$\begin{aligned}
\big|\psi_\pm^{(\pm)}\big|^2 &= \frac{\mathcal{P}_\perp \pm \mathcal{G}/2}{2\mathcal{P}_\perp}\big|\psi_\pm^{(1)}\big|^2 + \frac{\mathcal{P}_\perp \mp \mathcal{G}/2}{2\mathcal{P}_\perp}\big|\psi_\pm^{(2)}\big|^2\\
&= \frac{1}{6}\left[\sum_{m=-\infty}^{+\infty}\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}m\varphi}\exp(\mathrm{i}p_3 z)J_m(p_\perp r)\right]^2\\
&= F^2(r)\,.
\end{aligned} \tag{24}$$

This means that distributions of particles on the states with energy E_2 or E_1 are same and do not depend on the spin or color quantum numbers. This is a result of the invariance of the distribution with respect to the transformation (11):

$$\big|\psi_\pm^{(+)}\big|^2 + \big|\psi_\pm^{(-)}\big|^2 = \big|\psi_\pm^{(1)}\big|^2 + \big|\psi_\pm^{(2)}\big|^2.$$

As is known, eigenfunctions of any conserved quantity are eigenfunctions of the Hamiltonian as well. Now we wish to make clear what the conserved operator is in 6-dimensional combined spin–color spin space, the eigenfunctions of which are the found energy states $\psi_\pm^{(\pm)}$. Of course, the sought operator determines the branches of the spectrum and will commute with H'. In order to find this operator, let us divide the Hamiltonian (7) into a diagonal part $\mathbf{p}^2 + \frac{1}{2}g^2\tau I_2 + M^2$, which does not change under the transformation (12), and a non-diagonal part [19, 48],

$$\begin{aligned}
&\big(\sigma^i P_i\big)^2 - \left(\mathbf{p}^2 + \frac{1}{2}g^2\tau I_2\right)\\
&\quad = g\tau^{1/2}\left(p_1\lambda^1 + p_2\lambda^2 - \frac{1}{2}g\tau^{1/2}\sigma^3\lambda^3\right),
\end{aligned} \tag{25}$$

which becomes diagonal under this transformation:

$$\begin{aligned}
&\mathcal{U}^{-1}g\tau^{1/2}\left(p_1\lambda^1 + p_2\lambda^2 - \frac{1}{2}g\tau^{1/2}\sigma^3\lambda^3\right)\mathcal{U}\\
&= \begin{pmatrix} U_+^{-1} & 0\\ 0 & U_-^{-1}\end{pmatrix}\begin{pmatrix}(\lambda^b I^b)_+ & 0\\ 0 & (\lambda^b I^b)_-\end{pmatrix}\begin{pmatrix}U_+ & 0\\ 0 & U_-\end{pmatrix}\\
&= \begin{pmatrix}(\lambda^b I^b)_+^r & 0\\ 0 & (\lambda^b I^b)_-^r\end{pmatrix}.
\end{aligned} \tag{26}$$

This non-diagonal term is the interaction term of the particle with the chromomagnetic background. Here the two operators $(\lambda^b I^b)_\pm = g\tau^{1/2}\big(p_1\lambda^1 + p_2\lambda^2 \mp \frac{1}{2}g\tau^{1/2}\lambda^3\big)$ correspond to the two different spin states of the particle and in the transformed spin–color spin space these operators have got a diagonal form:

$$\begin{aligned}
(\lambda^b I^b)_+^r &= \begin{pmatrix}-\mathcal{G}\mathcal{P}_\perp & 0 & 0\\ 0 & \mathcal{G}\mathcal{P}_\perp & 0\\ 0 & 0 & 0\end{pmatrix},\\
(\lambda^b I^b)_-^r &= \begin{pmatrix}\mathcal{G}\mathcal{P}_\perp & 0 & 0\\ 0 & -\mathcal{G}\mathcal{P}_\perp & 0\\ 0 & 0 & 0\end{pmatrix}.
\end{aligned}$$

As is seen from (9) and (26), the term with the ± sign in the energy spectrum has appeared by this term, which can be written as the scalar product of two color vectors[3] λ^b and I^b:

$$\begin{aligned}
(\lambda^b I^b) &= g\lambda^b\left(A_j^b p_j - \frac{g}{4}\epsilon_{ij3}f^{acb}A_i^a A_j^c\sigma^3\right)\\
&= 2g\left(A_j p_j - \frac{1}{2}F_{12}\sigma^3\right).
\end{aligned} \tag{27}$$

Here $F_{12} = F_{12}^3\lambda^3/2$. This product is the projection of the color spin operator λ^b onto the color vector I^b, and in the rotated color space this projection has got the diagonal form with the three eigenvalues $\pm\mathcal{G}\mathcal{P}_\perp$, 0. The three different branches of the energy spectrum correspond to these three values of the $(\lambda^b I^b)$ projection. Since the operator $\sigma^3/2$ commutes with the general Hamiltonian (7),

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[3] By vector here we mean an eight component quantity in color space.

and consequently is a conserved quantity, this operator describes the spin of the particle in the field (4). The two Hamiltonians $H_\pm$ correspond to the two eigenvalues of this operator and each one in diagonalized form has three different eigenvalues E_1, E_2, E_3. Three different eigenfunctions $\psi_\pm^{(\pm),(3)}$ correspond to these eigenvalues of the spectrum. But the operator λ^3 does not commute with (7), and consequently it is not a conserved operator in spite of the field (4) being directed along the third axis in the color space as well. This is a reason why λ^3 does not determine the branches of the energy spectrum (9) and the projection of this operator onto the chromomagnetic field is not a suitable quantity for the description of the color states. A projection of the λ operator onto the color vector $\mathbf{I}$, i.e. the scalar product (27), commutes with the Hamiltonian (7), i.e., is a conserved quantity. As we see from this analysis, the term with the ± sign in the energy spectra (9) coming from the $(\lambda^b I^b)_\pm$ term in the Hamiltonians (6) in fact results in splitting of the energy spectrum into three branches, which correspond to the eigenvalues of this projection. The wave functions $\psi_\pm^{(\pm)}$ describing states with the energies from these branches of spectra are eigenfunctions of the operator (27) as well. Though the two different operators $(\lambda^b I^b)_-$ and $(\lambda^b I^b)_+$ correspond to the two different eigenvalues of the spin operator, the different eigenvalues of these operators coincide: $(\lambda^b I^b)_+^{(+)} = (\lambda^b I^b)_-^{(-)} = -\mathcal{G}\mathcal{P}_\perp$, $(\lambda^b I^b)_+^{(-)} = (\lambda^b I^b)_-^{(+)} = \mathcal{G}\mathcal{P}_\perp$, $(\lambda^b I^b)_+^{(3)} = (\lambda^b I^b)_-^{(3)} = 0$. Consequently, the wave functions of the states with the same spin projection, but with different value of $(\lambda^b I^b)^r$, describe the different energy branches: $\psi_+^{(+)} \to E_2$, $\psi_+^{(-)} \to E_1$ and $\psi_-^{(+)} \to E_1$, $\psi_-^{(-)} \to E_2$, and wave functions of the states with the same projection $(\lambda^b I^b)^r$, but with different spin projections, describe the different energy branches as well: $\psi_+^{(+)} \to E_2$, $\psi_-^{(+)} \to E_1$ and $\psi_+^{(-)} \to E_1$, $\psi_-^{(-)} \to E_2$. Since this operator has three eigenvalues instead of six, there arises a twofold degeneracy, which cannot be classified either as spin degeneracy or as degeneracy on the eigenvalues of the projection $(\lambda^b I^b)^r$. In the transformed space, where the operator $(\lambda^b I^b)$ has got its diagonal form,

$$(\lambda^b I^b)^r = \begin{pmatrix} -\mathcal{G}\mathcal{P}_\perp & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{G}\mathcal{P}_\perp & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{G}\mathcal{P}_\perp & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathcal{G}\mathcal{P}_\perp & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{28}$$

it can be written as the product of two operators, $-\sigma^3(\lambda^b I^b)_+^r$, and it is easily found that the degeneracy of the energy spectrum is the degeneracy on this product. Having expressed the spectrum branches in terms of the projections of σ^3 and $(\lambda^b I^b)_+^r$ we can establish the values of these projections, on which these branches coincide:

$$\begin{aligned} &E\bigl(\sigma^3 = 1\,;\, (\lambda^b I^b)^r = -\mathcal{G}\mathcal{P}_\perp\bigr) \\ &\quad = E\bigl(\sigma^3 = -1\,;\, (\lambda^b I^b)^r = \mathcal{G}\mathcal{P}_\perp\bigr) = E_2\,, \end{aligned}$$

$$\begin{aligned} &E\bigl(\sigma^3 = 1\,;\, (\lambda^b I^b)^r = \mathcal{G}\mathcal{P}_\perp\bigr) \\ &\quad = E\bigl(\sigma^3 = -1\,;\, (\lambda^b I^b)^r = -\mathcal{G}\mathcal{P}_\perp\bigr) = E_1\,. \end{aligned}$$

Thus we conclude, that this is a degeneracy on the quantity (27), which includes spin, color spin and momentum, in spite of there being no degeneracy on this quantities separately. As a result of this degeneracy the states having different quantum numbers σ^3 and $(\lambda^b I^b)^r$ have got the same energy: $\psi_+^{(+)}$, $\psi_-^{(-)} \to E_2$ and $\psi_-^{(+)}$, $\psi_+^{(-)} \to E_1$. After this analysis we come to idea to write the energy states as eigenfunctions of the $(\lambda^b I^b)_\pm^r$ operator, introducing unit eigenvectors $\zeta^{(\pm)}$ of these operators:

$$\begin{aligned} (\lambda^b I^b)_\pm^r \zeta^{(\pm)} &= \pm\mathcal{G}\mathcal{P}_\perp \lambda^3 \zeta^{(\pm)} = \pm\mathcal{G}\mathcal{P}_\perp (\pm\zeta^{(\pm)})\,, \\ (\lambda^b I^b)_+^r \zeta^{(3)} &= \mathcal{G}\mathcal{P}_\perp \lambda^3 \zeta^{(3)} = 0\,, \end{aligned} \tag{29}$$

which are basis vectors of the transformed color space as well:

$$\zeta^{(+)} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \zeta^{(-)} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \zeta^{(3)} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Then the energy states can be expressed on this basis as follows:

$$\psi_\pm^{(\pm)} = \zeta^{(\pm)} \xi_\pm F(r)\,. \tag{30}$$

As we noted, the term $(\lambda^b I^b)$ describes the interaction of the particle with the external field. This interaction occurs by means of the chromomagnetic moment of the particle due to its spin, which is included in the last term in (27) and by the chromomagnetic moment acquired due to the orbital moment, which we suppose is hidden in the first term of (27). In the Abelian theory, in the motion of an electron in an axial magnetic field given by the vector potential $A_\mu = \bigl(0\,,\, -\frac{1}{2}yH_z\,,\, \frac{1}{2}xH_z\,,\, 0\bigr)$, the term $A_j p_j$ in the Hamiltonian is written as the interaction term of the magnetic moment of electron due to the orbital moment L_z with the magnetic field H_z [11]:

$$\frac{e}{m_0 c} A_j p_j = -\frac{eH_z}{2m_0 c} L_z = -\frac{eH_z \hbar}{2m_0 c} m\,.$$

This term eliminates the degeneracy of the energy levels on the magnetic quantum number m and leads to the physical effect of splitting of the energy spectrum levels in the magnetic field. In our non-Abelian Landau problem the term $g\lambda^b A_j^b p_j = 2gA_j p_j$ cannot be written proportional to the orbital moment. It has the color matrix structure

$$\lambda^b A_j^b p_j = \mathcal{G}\begin{pmatrix} 0 & p_- & 0 \\ p_+ & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and acting of the $p_\pm$ on the wave functions $\mathrm{e}^{\mathrm{i}m\varphi} J_m(p_\perp r)$ is given by (22). In the quantized spectrum case $p_\perp = \alpha_m^{(N)}/r_0$ and for this interaction term we have the following eigenvalues:

$$\lambda^b A_j^b p_j \mathrm{e}^{\mathrm{i}m\varphi} J_m(p_\perp r) \begin{pmatrix} \xi_\pm^{(1)} \\ \xi_\pm^{(2)} \\ \xi_\pm^{(3)} \end{pmatrix} = \mathcal{G} \frac{\alpha_m^{(N)}}{r_0} \begin{pmatrix} -\mathrm{e}^{\mathrm{i}(m-1)m\varphi} J_{m-1}\big(\alpha_m^{(N)} r/r_0\big)\xi_\pm^{(2)} \\ \mathrm{e}^{\mathrm{i}(m+1)m\varphi} J_{m+1}\big(\alpha_m^{(N)} r/r_0\big)\xi_\pm^{(1)} \\ 0 \end{pmatrix}.$$

As is seen, the term $g\lambda^b A_j^b p_j$ in the Hamiltonian does not lead to the appearance of the terms with a factor of the chromomagnetic quantum number m and shifts the state with m to the states with $m \pm 1$. But this term contains $\alpha_m^{(N)}$, which depends on m and on N. This means that the term $g\lambda^b A_j^b p_j$ splits the spectrum into a series defined by N and the levels in the series are defined by m. Let us recall that the radii $a_m^{(N)}$ of the orbits, in which the particle rotates in this field, are also determined by $\alpha_m^{(N)}$, i.e. by both quantum numbers, m and N [7]: $a_m^{(N)} = m r_0/\alpha_m^{(N)}$. Thus we conclude that the interaction term $gA_j p_j$ in the non-Abelian theory, as a result of the interaction of the chromomagnetic moment due to the orbital moment with the chromomagnetic background, eliminates the degeneracy on m and N and splits the energy levels in the quantized spectrum case. Beside this splitting, this interaction shifts the particle from the state m to the states $m \pm 1$ and consequently from the orbit $a_m^{(N)}$ to the orbits $a_{m\pm1}^{(N)}$ in dependence of the projection of λ^3 onto the chromomagnetic field. In the transformed color space, according to (29) this interaction term splits the spectrum into branches, and each branch contains a factor $\mathcal{G}P_\perp = \mathcal{G}\sqrt{p_\perp^2 + \mathcal{G}^2/4} = \mathcal{G}\sqrt{(\alpha_m^{(N)}/r_0)^2 + \mathcal{G}^2/4}$, which splits the spectrum branches into the series.

3 Spherical background

Let us consider the case of a constant spherical chromomagnetic field defined by the constant vector potential:

$$A_1^a = \sqrt{\tau}\delta_{1a}\,, \quad A_2^a = \sqrt{\tau}\delta_{2a}\,, \quad A_3^a = \sqrt{\tau}\delta_{3a}\,, \quad A_0^a = 0\,. \tag{31}$$

This field has a strength tensor with the following non-zero components:

$$F_{23}^1 = F_{31}^2 = F_{12}^3 = H_x^1 = -H_y^2 = H_z^3 = g\tau \tag{32}$$

and possesses spherical symmetry in ordinary space and in the subspace of the first three coordinates in the color space. The squared Dirac equation (2) in the field (31) has got the following explicit form:

$$\left(\mathbf{p}^2 + M^2 + \frac{3g^2\tau}{4} + g\tau^{1/2}\lambda^a p^a - \frac{g^2\tau}{2}\sigma^a\lambda^a\right)\mathit{\Psi} = E^2\mathit{\Psi}\,. \tag{33}$$

The spinor $\mathit{\Psi}$ has the two components ψ_+ and ψ_-, and each of them transforms under the fundamental representation of the $SU_c(3)$ color group. Because the external field (32) has the three non-zero components in both spaces, in this case the spin and color spin states of the particle cannot be described by the projection of the σ^3 and λ^3 operators. So the components $\psi_\pm^{(a)}$ are not eigenfunctions of these operators and do not describe states with definite values of their projections onto the field.

Equation (33) contains the non-diagonal σ^a and λ^a matrices, which make it non-diagonal. As a result, this equation does not split into independent ones for the components $\psi_\pm^{(a)}$. From (33) can be obtained the determinant equation, which is the same for all states $\psi_j^{(i)}$ $(i = 1, 2)$ and has the following solution [7]:

$$\psi_j^{(i)}(\mathbf{r}) = \frac{C_l^\nu}{|\mathbf{p}|\sqrt{r}} J_{l+1/2}(|\mathbf{p}|r) Y_l^m(\theta, \varphi)\xi_j^{(i)}, \tag{34}$$

where $J_{l+1/2}(|\mathbf{p}|r)$ is a Bessel function of the first kind, $Y_l^m(\theta, \varphi)$ are spherical functions, C_l^ν are normalizing constants ($C_l^\nu = |\mathbf{p}|$), and the $\xi_j^{(i)}$ are the spin and color spin parts of the wave function. Solving (33) for the energy gives the following branches of the continuous spectrum in the infinite motion case [14–17]:

$$\begin{aligned} E_{1,2}^2 &= \left(\sqrt{\mathbf{p}^2} \mp \frac{\mathcal{G}}{2}\right)^2 + M^2, \\ E_{3,4}^2 &= \left(\sqrt{\mathbf{p}^2 + \mathcal{G}^2} \mp \frac{\mathcal{G}}{2}\right)^2 + M^2, \\ E_5^2 &= \mathbf{p}^2 + M^2, \end{aligned}$$

and the quantized spectrum branches in the case of the finite motion inside the sphere [7] are

$$\begin{aligned} E_{1,2}^2 &= \left(\frac{\alpha_l^{(N)}}{r_0} \mp \frac{\mathcal{G}}{2}\right)^2 + M^2, \\ E_{3,4}^2 &= \left(\sqrt{\left(\frac{\alpha_l^{(N)}}{r_0}\right)^2 + \mathcal{G}^2} \mp \frac{\mathcal{G}}{2}\right)^2 + M^2, \\ E_5^2 &= \left(\frac{\alpha_l^{(N)}}{r_0}\right)^2 + M^2. \end{aligned} \tag{35}$$

Here $\alpha_l^{(N)}$ are the zeros of $J_{l+1/2}(|\mathbf{p}|r)$, r_0 is the radius of the sphere, and l is the orbital quantum number. Quantized and continuous spectra are interrelated by means of the replacement $|\mathbf{p}| = \alpha_l^{(N)}/r_0$, which is the same as quantization of the momentum in standing waves. Motion in this case takes place on s, p, d, f, … orbitals. As we observe, the additional non-zero components of the field strength tensor F_{23}^1 and F_{31}^2 apparently break the remaining color and spin symmetry in the Hamiltonian. As a result of this breakdown the spectrum splits into four branches as distinct from the axial background field case, i.e. leads to elimination of the degeneracy of the energy spectrum. From the spectrum (35) is also seen that the energy E_5 does not contain the interaction term with the external field and so does not split into branches. The state $\psi_j^{(3)}(\mathbf{r})$ is like the state of a spinless and colorless particle. But solving (33) by means of its determinant leads to the absence of a correspondence between the branches of spectrum (35) and the solutions (34). Consequently, it is reasonable to find the wave functions, which will describe the states having a definite energy from these branches, i.e. to find the energy states in the field (31).

Let us write the explicit matrix form of the Hamiltonian for (33) in the spin–color spin space:

$$H = \begin{pmatrix} h_{11} & \mathcal{G}p_- & 0 & 0 & 0 & 0 \\ \mathcal{G}p_+ & h_{22} & 0 & -\mathcal{G}^2 & 0 & 0 \\ 0 & 0 & h_{33} & 0 & 0 & 0 \\ 0 & -\mathcal{G}^2 & 0 & h_{44} & \mathcal{G}p_- & 0 \\ 0 & 0 & 0 & \mathcal{G}p_- & h_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & h_{66} \end{pmatrix} . \tag{36}$$

The diagonal elements h_{ii} have the expressions

$$\begin{aligned} &h_{11} = \mathcal{P}^2 + \mathcal{G}^2/4 + \mathcal{G}p_3 \,, && h_{22} = \mathcal{P}^2 + 5\mathcal{G}^2/4 - \mathcal{G}p_3 \,, \\ &h_{33} = \mathcal{P}^2 \,, && h_{44} = \mathcal{P}^2 + 5\mathcal{G}^2/4 + \mathcal{G}p_3 \,, \\ &h_{55} = \mathcal{P}^2 + \mathcal{G}^2/4 - \mathcal{G}p_3 \,, && h_{66} = \mathcal{P}^2 \,, \end{aligned} \tag{37}$$

where $\mathcal{P}^2 = \mathbf{p}^2 + M^2$. The Hamiltonian (36) does not have a quasidiagonal form and so we cannot diagonalize separately the two matrices $H_\pm$ with dimensions 3×3 instead of one with dimensions 6×6. In order to diagonalize (36) we should make the transformation in the combined space of spin and color spin. We can find this transformation and the diagonal form of (36) applying the method used in the previous section.

As was asserted in the previous section, the diagonal form of H' is unique and the transformation U transforming H into H' is unique and unitary due hermiticity of H. Relying on these two properties, we can find the matrices H' and $\mathcal{U}$ simultaneously. From the beginning we find the determinant of this matrix, which equals

$$\begin{aligned} \det H &= (\mathcal{P}^2)^2((\mathcal{P}^2 + \mathcal{G}^2/4)^2 - \mathcal{G}^2 p^2) \\ &\quad \times ((\mathcal{P}^2 + 5\mathcal{G}^2/4)^2 - \mathcal{G}^2 p'^2) \\ &= \det H' \end{aligned} \tag{38}$$

and can be written as a product of six factors f_i:

$$\begin{aligned} &f_{1,2} = \mathcal{P}^2 \,, \quad f_{3,4} = \mathcal{P}^2 + \mathcal{G}^2/4 \pm \mathcal{G}p \,, \\ &f_{5,6} = \mathcal{P}^2 + 5\mathcal{G}^2/4 \pm \mathcal{G}p', \end{aligned}$$

where $p = |\mathbf{p}| = \sqrt{\mathbf{p}^2}$, $p' = \sqrt{\mathbf{p}^2 + \mathcal{G}^2}$. The sum of these factors equals the trace of the Hamiltonian (36):

$$\sum_i f_i = 6\mathcal{P}^2 + 3\mathcal{G}^2 = \mathrm{Tr} H = \sum_i h_{ii} = \sum_i h'_{ii} \,.$$

According to the invariance of the trace and determinant of the matrix under the unitary transformation, we may suppose that the factors f_i are the diagonal elements of the diagonalized Hamiltonian H'. But this is not enough to find the explicit form of H', as we do not know the place of each of factors f_i along the diagonal of H' and have no rule to determine this place. At first, we can assume some identification of factors f_i with the diagonal elements of H' and then write (13) explicitly for this assumption with unknown $\mathcal{U}$. The equations for u_{ij} obtained from (13) together with the equations obtained from the unitarity condition will be solved regularly, if the diagonal elements were identified properly. If the factors f_i corresponded to the diagonal elements h'_{ii} incorrectly, then solving (13) will lead to mathematical nonsense due to the uniqueness of H'. So we make the following identification:

$$\begin{aligned} h'_{11} &\equiv f_4 = \mathcal{P}^2 + g^2\tau/4 - \mathcal{G}p \,, \\ h'_{22} &\equiv f_6 = \mathcal{P}^2 + 5g^2\tau/4 - \mathcal{G}p' \,, \\ h'_{33} &\equiv f_1 = \mathcal{P}^2 \,, \\ h'_{44} &\equiv f_5 = \mathcal{P}^2 + 5g^2\tau/4 + \mathcal{G}p' \,, \\ h'_{55} &\equiv f_3 = \mathcal{P}^2 + g^2\tau/4 + \mathcal{G}p \,, \\ h'_{66} &\equiv f_2 = \mathcal{P}^2 \,. \end{aligned} \tag{39}$$

Writing (13) for this choice, we get systems of linear equations relating the elements u_{ij} of the transformation matrix $\mathcal{U}$, which are simplified into the following form:

$$\begin{aligned} &\begin{cases} u_{11} = u_{21} p_- / (p - p_3) \,, \\ u_{41} = u_{21} \,, \\ u_{51} = u_{21} p_+ / (p + p_3) \,, \end{cases} \\ &\begin{cases} u_{12} = -u_{22} p_- / \big(p' + p_3 - \mathcal{G}\big) \,, \\ u_{42} = -u_{22} \big(p' - p_3 - \mathcal{G}\big) / \big(p' + p_3 - \mathcal{G}\big) \,, \\ u_{52} = -u_{22} p_+ / \big(p' + p_3 - \mathcal{G}\big) \,, \end{cases} \\ &\begin{cases} u_{14} = u_{24} p_- / \big(p' - p_3 + \mathcal{G}\big) \,, \\ u_{44} = -u_{24} \big(p' + p_3 + \mathcal{G}\big) / \big(p' - p_3 + \mathcal{G}\big) \,, \\ u_{54} = -u_{24} p_+ / \big(p' - p_3 + \mathcal{G}\big) \,, \end{cases} \\ &\begin{cases} u_{15} = -u_{25} p_- / (p + p_3) \,, \\ u_{45} = u_{25} \,, \\ u_{55} = -u_{25} p_+ / (p - p_3) \,. \end{cases} \end{aligned} \tag{40}$$

All other u_{ij} are zero except for the u_{33}, u_{36}, u_{63} and u_{66} elements.

All other elements u_{ij} in (40) are expressed in terms of u_{2j}. The equations (40) relate 32 unknowns by 24 relations. So in order to find all u_{ij} we need additional relations. For this purpose we can use relations of the unitarity condition, which are six equations $\sum_{j=1}^{6} u_{ij} u_{ij}^* = 1$ and 30 ones $\sum_{j=1}^{6} u_{ij} u_{kj}^* = 0$, but these relations are nonlinear. Besides, some of the unitarity relations coincide with other ones after taking into account the relations of (40) in them. Thus, taking into account the relations of (40) in the unitary relations and solving them we find only the modulus of the u_{2i}:

$$\begin{aligned} &u_{21} u_{21}^* = \frac{p_\perp^2}{4p^2} \,, && u_{22} u_{22}^* = \frac{\big(p' + p_3 - \mathcal{G}\big)^2}{4p'(p' - \mathcal{G})} \,, \\ &u_{24} u_{24}^* = \frac{\big(p' + p_3 - \mathcal{G}\big)^2}{4p'(p' + \mathcal{G})} \,, && u_{25} u_{25}^* = \frac{p_\perp^2}{4p^2} \,. \end{aligned} \tag{41}$$

The arguments of u_{2i} remain unknown and we parametrize them introducing the free angles α, β, γ and δ as follows:

$$\begin{aligned} u_{21} &= \frac{p_\perp}{2p}\mathrm{e}^{\mathrm{i}\alpha}, & u_{22} &= \frac{p'+p_3-\mathcal{G}}{[4p'(p'-\mathcal{G})]^{1/2}}\mathrm{e}^{\mathrm{i}\beta}, \\ u_{24} &= \frac{p'+p_3-\mathcal{G}}{[4p'(p'+\mathcal{G})]^{1/2}}\mathrm{e}^{\mathrm{i}\gamma}, & u_{25} &= \frac{p_\perp}{2p}\mathrm{e}^{\mathrm{i}\delta}. \end{aligned} \tag{42}$$

In the system (40) there is no equation relating these four elements and so we cannot express one angle in terms of the other ones[4]. As the elements u_{3j}, u_{j3}, u_{6j} and u_{j6} are equal to zero, (13) gives trivial relations for the elements u_{33}, u_{36}, u_{63} and u_{66} and does not relate them with any other u_{ij} in (40). The relations for these elements, obtained from the unitarity condition, consist of the following ones:

$$\begin{cases} u_{33}u_{33}^* + u_{36}u_{36}^* = 1\,, \\ u_{63}u_{63}^* + u_{66}u_{66}^* = 1\,, \\ u_{33}u_{63}^* + u_{36}u_{66}^* = 0\,. \end{cases} \tag{43}$$

The relations in (43) are nonlinear and their number is less than the number of unknowns. Consequently, we get one more free parameter introducing the angle θ in order to parametrize the elements in (43):

$$\begin{aligned} u_{33} &= \cos\theta\,, & u_{36} &= \sin\theta\,, \\ u_{63} &= \sin\theta\,, & u_{66} &= -\cos\theta\,. \end{aligned}$$

It is reasonable to assume that under the $\mathcal{U}$ transformation the states $\psi_\pm^{(3)}$ do not change, since these states are like colorless states ($\lambda^3 = 0$). Under this assumption we can fix the parameter $\theta = 0$ and then $u_{33} = -u_{66} = 1$, $u_{36} = u_{63} = 0$.

As has been expressed before, the Hamiltonian (36) owing to the non-zero h_{42} and h_{24} elements has no quasidiagonal form and mixes different spin states. So we cannot demand the unimodularity of any block of the $\mathcal{U}$ matrix, as we did in the previous case. We also have no additional conditions to cut down the number of free parameters (α, β, γ, δ) or fix them. Thus, the $\mathcal{U}$ matrix contains four free parameters and has the following explicit form:

$$\mathcal{U} = \begin{pmatrix} p_-\mathcal{P}_\perp/\mathcal{P}_3 & -p_-/\mathcal{P}_1 & 0 & p_-/\mathcal{P}_2 & p_-\mathcal{P}'_\perp/\mathcal{P}_4 & 0 \\ \mathcal{P}_\perp & (p'+p_3-\mathcal{G})/\mathcal{P}_1 & 0 & (p'-p_3+\mathcal{G})/\mathcal{P}_2 & \mathcal{P}'_\perp & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{P}_\perp & (p'-p_3-\mathcal{G})/\mathcal{P}_1 & 0 & -(p'+p_3+\mathcal{G})/\mathcal{P}_2 & \mathcal{P}'_\perp & 0 \\ p_+\mathcal{P}_\perp/\mathcal{P}_4 & p_+/\mathcal{P}_1 & 0 & -p_+/\mathcal{P}_2 & p_+\mathcal{P}'_\perp/\mathcal{P}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{44}$$

where $\mathcal{P}_1 = [4p'(p'-\mathcal{G})]^{1/2}\mathrm{e}^{-\mathrm{i}\beta}$, $\mathcal{P}_2 = [4p'(p'+\mathcal{G})]^{1/2}\mathrm{e}^{-\mathrm{i}\gamma}$, $\mathcal{P}_{3,4} = p \mp p_3$, $\mathcal{P}_\perp = (p_\perp/2p)\mathrm{e}^{\mathrm{i}\alpha}$, $\mathcal{P}'_\perp = -(p_\perp/2p)\mathrm{e}^{\mathrm{i}\delta}$. Transforming the basis vectors $\psi_j^{(i)}$ under the $\mathcal{U}$ transformation, we can construct the energy states according to (20). Multiplying the $\mathcal{U}$ matrix (44) by the column of solutions (34) we find the energy states:

$$\begin{aligned} \psi_+^{(+)} &= \frac{p_-\mathcal{P}_\perp}{\mathcal{P}_3}\psi_1^{(1)}(\mathbf{r}) - \frac{p_-}{\mathcal{P}_1}\psi_1^{(2)}(\mathbf{r}) \\ &\quad + \frac{p_-}{\mathcal{P}_2}\psi_2^{(1)}(\mathbf{r}) + \frac{p_-\mathcal{P}'_\perp}{\mathcal{P}_4}\psi_2^{(2)}(\mathbf{r})\,, \\ \psi_+^{(-)} &= \mathcal{P}_\perp\psi_1^{(1)}(\mathbf{r}) + \frac{(p'+p_3-\mathcal{G})}{\mathcal{P}_1}\psi_1^{(2)}(\mathbf{r}) \\ &\quad + \frac{(p'-p_3+\mathcal{G})}{\mathcal{P}_2}\psi_2^{(1)}(\mathbf{r}) + \mathcal{P}'_\perp\psi_2^{(2)}(\mathbf{r})\,, \\ \psi_-^{(+)} &= \mathcal{P}_\perp\psi_1^{(1)}(\mathbf{r}) + \frac{(p'-p_3-\mathcal{G})}{\mathcal{P}_1}\psi_1^{(2)}(\mathbf{r}) \\ &\quad - \frac{(p'+p_3+\mathcal{G})}{\mathcal{P}_2}\psi_2^{(1)}(\mathbf{r}) + \mathcal{P}'_\perp\psi_2^{(2)}(\mathbf{r})\,, \\ \psi_-^{(-)} &= \frac{p_+\mathcal{P}_\perp}{\mathcal{P}_4}\psi_1^{(1)}(\mathbf{r}) + \frac{p_+}{\mathcal{P}_1}\psi_1^{(2)}\mathbf{r} \\ &\quad - \frac{p_+}{\mathcal{P}_2}\psi_2^{(1)}(\mathbf{r}) + \frac{p_+\mathcal{P}'_\perp}{\mathcal{P}_3}\psi_2^{(2)}(\mathbf{r})\,. \end{aligned} \tag{45}$$

We can replace all momentum operators in (45) by their eigenvalues. This does not lead to any changes in (45). Remark that the transformation $\mathcal{U}$ acts on the 6-dimensional combined space of spin and color spin, so the up and down ± signs in $\psi_\pm^{(\pm)}$ in (45) are just only a matter of notation and have no meaning as projection signs.

Comparing the spectrum branches (34) with the diagonal elements of H' (39), we write down the obvious correspondence between them:

$$\begin{aligned} E_1 &\longrightarrow h'_{11}\,, \\ E_2 &\longrightarrow h'_{55}\,, \\ E_3 &\longrightarrow h'_{22}\,, \\ E_4 &\longrightarrow h'_{44}\,. \end{aligned}$$

Using this correspondence we can write the eigenvalue equation (10) for H' by the eigenfunctions $\psi_\pm^{(\pm)}$ (45) and the spectrum (35):

$$\begin{pmatrix} h'_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & h'_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h'_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & h'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & h'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & h'_{66} \end{pmatrix} \begin{pmatrix} \psi_+^{(+)} \\ \psi_+^{(-)} \\ \psi_+^{(3)} \\ \psi_-^{(+)} \\ \psi_-^{(-)} \\ \psi_-^{(3)} \end{pmatrix} = \begin{pmatrix} E_1^2\psi_+^{(+)} \\ E_3^2\psi_+^{(-)} \\ E_5^2\psi_+^{(3)} \\ E_4^2\psi_-^{(+)} \\ E_2^2\psi_-^{(-)} \\ E_5^2\psi_-^{(3)} \end{pmatrix}.$$

Thus, we find the wave functions of a colored particle describing the states having the energy from the spectrum (34). This can be written in the following correspondence between the states (45) and branches of the energy spectrum (34):

$$\begin{aligned} E_1 &\longrightarrow \psi_+^{(+)}\,, \\ E_2 &\longrightarrow \psi_-^{(-)}\,, \\ E_3 &\longrightarrow \psi_+^{(-)}\,, \\ E_4 &\longrightarrow \psi_-^{(+)}\,. \end{aligned}$$

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[4] This was done in the previous case.

Consider now the operator which determines the branches of the spectra (34), i.e., separate the term in Hamiltonian describing the interaction of the particle with the external field. The non-diagonal part of the Hamiltonian (36),

$$g\tau^{1/2}\lambda^b\left(p^b - \frac{g\tau^{1/2}}{2}\sigma^b\right) = g\lambda^b\left(A^b_j p_j - \frac{g}{4}\epsilon_{ijk} f^{acb} A^a_i A^c_j \sigma^k\right) = (\lambda^b I^b), \tag{46}$$

under the transformation (12) with the $\mathcal{U}$ matrix (44) transforms it to diagonal form:

$$(\lambda^b I^b)^r = \mathcal{U}^{-1}(\lambda^b I^b)\mathcal{U} = \begin{pmatrix} -\mathcal{G}p & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathcal{G}p' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{G}p' & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{G}p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

Here ϵ_{ijk} is the unit antisymmetric tensor. As is seen from (47), in this case the diagonal form of the operator (47) has five different eigenvalues, each one of which corresponds to the one branch of the energy spectrum (34). In the non-transformed space this operator can also be written in terms of the interactions of the chromomagnetic moments with the chromomagnetic background:

$$(\lambda^b I^b) = g\lambda^b\left(A^b_j p_j - \frac{1}{2}\epsilon_{ijk} F^b_{ij}\sigma^k\right) = 2g\left(A_j p_j - \frac{1}{2}\epsilon_{ijk} F_{ij}\sigma^k\right). \tag{48}$$

The second term in (48) describes the interaction of the particle with the external field due to its spin, and first one hides the interaction with the external field due to the orbital moment. In the transformed space these two kinds of interactions are joined in the eigenvalues $\pm\mathcal{G}p$, $\pm\mathcal{G}p'$ and 0 and cannot be separated. The operator $(\lambda^b I^b)$ commutes with the Hamiltonian (36) and the quantity corresponding to it is conserved. Thus we conclude that in external fields, given by the non-commuting potentials (4) and (31), the projection $(\lambda^b I^b)$ describes the interaction of the chromomagnetic moment of the particle with the external chromomagnetic field and causes the splitting of the energy spectrum into the branches and levels. States with a definite energy in these fields are determined by this projection instead of the projection of spin and color spin onto the field, i.e. are defined in that basis, in which this projection has got the diagonal form.

4 Superpartner states

As we know [18,19], (2) in the field (5) possesses supersymmetry, i.e., for the motion with $p_3 = 0$ the Hamiltonian $H_1 = H - M^2$ and the two operators $Q_\pm = P_\mp a_\pm$, which are named the supercharge operators, form a supersymmetry algebra in quantum mechanics [41–47]:

$$\{Q_+, Q_-\} = H_1\,, \quad [Q_\pm, H_1] = 0\,, \quad Q_+^2 = Q_-^2 = 0\,. \tag{49}$$

Here the braces denote an anticommutator and the square brackets denote a commutator. Defined by the formulae

$$P_\pm = P_1 \pm \mathrm{i}P_2\,, \quad a_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2) \tag{50}$$

the operators $P_\pm$ and $a_\pm$ obey the following commutation and anticommutation relations for the annihilation and creation operators of bosonic and fermionic states, respectively:

$$[P_+, P_-] = \lambda^3 g H_z^3\,, \quad \{a_+, a_-\} = 1\,. \tag{51}$$

Let us remark that the operators for creation and annihilation of the bosonic states interchange roles for the state having the value $\lambda^3 = 1$ with the state having the value $\lambda^3 = -1$ [19], which can be seen from (51). The action of the fermionic operators $a_\pm$ turns over the spin of the particle:

$$a_+\xi_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \xi_+\,, \quad a_+\xi_+ = 0\,;$$
$$a_-\xi_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \xi_-\,, \quad a_-\xi_- = 0\,. \tag{52}$$

Since the action of a_+ on $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ gives ψ_- and a_- gives ψ_+, the bosonic creation operator P_+ acts only on the ψ_+ state and the annihilation operator P_- only on the ψ_- state:

$$P_+\begin{pmatrix} \psi_+^{(1)} \\ \psi_+^{(2)} \\ \psi_+^{(3)} \end{pmatrix} = \begin{pmatrix} p_+ & \mathcal{G} & 0 \\ 0 & p_+ & 0 \\ 0 & 0 & p_+ \end{pmatrix}\begin{pmatrix} \psi_+^{(1)} \\ \psi_+^{(2)} \\ \psi_+^{(3)} \end{pmatrix} = \begin{pmatrix} p_+\psi_-^{(1)} + \mathcal{G}\psi_-^{(2)} \\ p_+\psi_-^{(2)} \\ p_+\psi_-^{(3)} \end{pmatrix},$$
$$P_-\begin{pmatrix} \psi_-^{(1)} \\ \psi_-^{(2)} \\ \psi_-^{(3)} \end{pmatrix} = \begin{pmatrix} p_- & 0 & 0 \\ \mathcal{G} & p_- & 0 \\ 0 & 0 & p_- \end{pmatrix}\begin{pmatrix} \psi_-^{(1)} \\ \psi_-^{(2)} \\ \psi_-^{(3)} \end{pmatrix} = \begin{pmatrix} p_-\psi_+^{(1)} \\ \mathcal{G}\psi_+^{(1)} + p_-\psi_+^{(2)} \\ p_-\psi_+^{(3)} \end{pmatrix}. \tag{53}$$

Using the supercharge operators it is easy to observe the spin diagonal form of the Hamiltonian H_1:

$$H_1\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} P_-P_+ & 0 \\ 0 & P_+P_- \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{54}$$

Note that the actions of the bosonic creation and annihilation operators $P_\pm$ mix the different color states $\psi^{(1)}$ and $\psi^{(2)}$. In supersymmetric quantum mechanics each quantum mechanical state of a particle is labeled by fermionic and bosonic quantum numbers [41–44]. States having the same energy are called superpartner states in this theory, but they have different fermionic and bosonic quantum numbers. The action of the supercharge operators $Q_\pm$

change the fermionic and bosonic quantum numbers and convert the superpartner states into each other.

In the first section we have established that energy states having different spin projections, namely the states $\psi_+^{(-)}$ and $\psi_-^{(+)}$ and the states $\psi_+^{(+)}$ and $\psi_-^{(-)}$, have the same energies. So we have a reason to look for superpartner states among the states $\psi_+^{(\pm)}$ and $\psi_-^{(\pm)}$. Since these states differ by the spin projection we can assert that they are turned into each other under the action of the fermionic operators $a_\pm$. If we find a couple of mutually hermitian bosonic operators $P_\pm$, which obey (51) and (54), and change the upper index of $\psi^{(\pm)}$, i.e. act by the rule $P_\pm\psi^{(\pm)} \to \psi^{(\mp)}$, then we can construct the supercharge operators $Q_\pm$ by means of these operators, and they will obey the supersymmetry algebra (49) and will map $\psi_\pm^{(\pm)}$ into each other:

$$\begin{aligned} Q_+\psi_-^{(-)} &= q_1\psi_+^{(-)}, \qquad Q_+\psi_-^{(-)} = q_2\psi_+^{(+)};\\ Q_-\psi_+^{(-)} &= q_1'\psi_-^{(+)}, \qquad Q_-\psi_+^{(+)} = q_2'\psi_-^{(-)}. \end{aligned} \tag{55}$$

The states $\psi_+^{(-)}$ and $\psi_-^{(+)}$ and the states $\psi_+^{(+)}$ and $\psi_-^{(-)}$ will be superpartner states in the framework of this supersymmetry. So for the superpartnership of these states it is enough to find suitable bosonic operators.

Under the $\mathcal{U}$ transformation the operators $P_\pm$ transform as well:

$$\begin{aligned} U_\pm^{-1}H_\pm U_\pm &= U_\pm^{-1}P_\mp P_\pm U_\pm\\ &= U_\pm^{-1}P_\mp U_\pm U_\pm^{-1}P_\pm U_\pm = P_\mp' P_\pm'\,. \end{aligned} \tag{56}$$

Applying the $U_\pm$ transformation we find the $P_\pm'$ operators in the new basis:

$$\begin{aligned} P_+' &= \frac{1}{2\mathcal{P}_\perp}\\ &\times \begin{pmatrix} 2p_+(\mathcal{P}_\perp+\mathcal{G}/2) & \mathcal{G}(\mathcal{P}_\perp+\mathcal{G}/2)\mathrm{e}^{-2\mathrm{i}\alpha} & 0\\ -\mathcal{G}p_+^2(\mathcal{P}_\perp+\mathcal{G}/2)^{-1}\mathrm{e}^{2\mathrm{i}\alpha} & 2p_+(\mathcal{P}_\perp-\mathcal{G}/2) & 0\\ 0 & 0 & 2\mathcal{P}_\perp p_+ \end{pmatrix},\\ P_-' &= \frac{1}{2\mathcal{P}_\perp}\\ &\times \begin{pmatrix} 2p_-(\mathcal{P}_\perp+\mathcal{G}/2) & -\mathcal{G}p_-^2(\mathcal{P}_\perp+\mathcal{G}/2)^{-1}\mathrm{e}^{-2\mathrm{i}\alpha} & 0\\ \mathcal{G}(\mathcal{P}_\perp+\mathcal{G}/2)\mathrm{e}^{2\mathrm{i}\alpha} & 2p_-(\mathcal{P}_\perp-\mathcal{G}/2) & 0\\ 0 & 0 & 2\mathcal{P}_\perp p_- \end{pmatrix}. \end{aligned} \tag{57}$$

The products of the transformed bosonic operators are the diagonalized Hamiltonians $H_\pm'$ (17) and (26) with $M^2 = 0$ and $p_3 = 0$:

$$P_-'P_+' = H_+'\,, \quad P_+'P_-' = H_-'\,. \tag{58}$$

It is clear that the new supercharge operators $Q_\pm' = P_\mp' a_\pm$ will obey the supersymmetry algebra (49) with the diagonalized Hamiltonian H':

$$\{Q_+', Q_-'\} = H_1'\,, \quad [Q_\pm', H_1'] = 0\,, \quad Q_+'^2 = Q_-'^2 = 0\,. \tag{59}$$

But we see from the explicit form of $P_\pm'$ that the action of these operators on $\varPsi' = \begin{pmatrix}\psi_\pm^{(+)}\\ \psi_\pm^{(-)}\\ \psi_\pm^{(3)}\end{pmatrix}$ does not map the states $\psi^{(+)}$ and $\psi^{(-)}$ into each other and mixes these states. This means that in the result of the action of these operators we get a superposition of the $\psi^{(\pm)}$ states, i.e. states having an uncertain energy. So the operators $P_\pm'$ are unusable to build a supersymmetry algebra in which the states $\psi^{(+)}$ and $\psi^{(-)}$ would turn out to be superpartners. But, fortunately, there can be constructed another mutually hermitian conjugate couple of $P_\pm''$ operators:

$$\begin{aligned} P_+'' &= \mathcal{P}_\perp\lambda^1 + \mathrm{i}\lambda^2\mathcal{G}/2 + p_+(I_3 - I_2)\\ &= \begin{pmatrix} 0 & \mathcal{P}_\perp+\mathcal{G}/2 & 0\\ \mathcal{P}_\perp-\mathcal{G}/2 & 0 & 0\\ 0 & 0 & p_+ \end{pmatrix},\\ P_-'' &= \mathcal{P}_\perp\lambda^1 - \mathrm{i}\lambda^2\mathcal{G}/2 + p_-(I_3 - I_2)\\ &= \begin{pmatrix} 0 & \mathcal{P}_\perp-\mathcal{G}/2 & 0\\ \mathcal{P}_\perp+\mathcal{G}/2 & 0 & 0\\ 0 & 0 & p_- \end{pmatrix}, \end{aligned} \tag{60}$$

which obey the commutation relation $[P_+'', P_-''] = 2\lambda^3\mathcal{G}\mathcal{P}_\perp$. The product of these operators also gives $H_\pm'$:

$$P_-''P_+'' = H_+'\,, \quad P_+''P_-'' = H_-'$$

and so the supercharge operators constructed using $Q_\pm' = P_\mp'' a_\pm$ also obey the supersymmetry algebra (59). The action of the $P_\pm''$ operators on $\varPsi'$ change the upper index of the components of this wave function:

$$\begin{aligned} P_+''\begin{pmatrix}\psi^{(+)}\\ \psi^{(-)}\\ \psi^{(3)}\end{pmatrix} &= \begin{pmatrix}(\mathcal{P}_\perp+\mathcal{G}/2)\psi^{(-)}\\ (\mathcal{P}_\perp-\mathcal{G}/2)\psi^{(+)}\\ p_+\psi^{(3)}\end{pmatrix},\\ P_-''\begin{pmatrix}\psi^{(-)}\\ \psi^{(+)}\\ \psi^{(3)}\end{pmatrix} &= \begin{pmatrix}(\mathcal{P}_\perp-\mathcal{G}/2)\psi^{(+)}\\ (\mathcal{P}_\perp+\mathcal{G}/2)\psi^{(-)}\\ p_-\psi^{(3)}\end{pmatrix}. \end{aligned} \tag{61}$$

The action of these operators on $\zeta^{(\pm),3}$ is given by the formulae

$$\begin{aligned} &P_+''\zeta^{(+)} = (\mathcal{P}_\perp-\mathcal{G}/2)\zeta^{(-)}, \quad P_+''\zeta^{(-)} = (\mathcal{P}_\perp+\mathcal{G}/2)\zeta^{(+)};\\ &P_-''\zeta^{(+)} = (\mathcal{P}_\perp+\mathcal{G}/2)\zeta^{(-)}, \quad P_-''\zeta^{(-)} = (\mathcal{P}_\perp-\mathcal{G}/2)\zeta^{(+)};\\ &P_\pm''\zeta^{(3)} = p_\pm\zeta^{(3)}. \end{aligned} \tag{62}$$

So this action changes the orientation of the I^b vector in the transformed color space. The operators $b_\pm = P_\pm''/(2\mathcal{G}\mathcal{P}_\perp)^{1/2}$ obey the Heisenberg–Weyl algebra

$$[b_+, b_-] = 1$$

for the state $\zeta^{(+)}$, i.e., they are the creation and annihilation operators, respectively. For the state $\zeta^{(-)}$ these operators obey the commutation relation

$$[b_-, b_+] = 1\,,$$

which again means interchanging the roles of the creation and annihilation operators of the $\zeta^{(+)}$ state.

We can write the action of the $Q'_\pm$ operators on $\psi_\pm^{(\pm)}$ using (62), (52) and (30):

$$\begin{aligned} Q'_+\psi_-^{(+)} &= P''_- a_+ \xi_- \zeta^{(+)} F(r) = (\mathcal{P}_\perp + \mathcal{G}/2)\psi_+^{(-)}, \\ Q'_+\psi_-^{(-)} &= P''_- a_+ \xi_- \zeta^{(-)} F(r) = (\mathcal{P}_\perp - \mathcal{G}/2)\psi_+^{(+)}, \\ Q'_-\psi_+^{(-)} &= P''_+ a_- \xi_+ \zeta^{(-)} F(r) = (\mathcal{P}_\perp + \mathcal{G}/2)\psi_-^{(+)}, \\ Q'_-\psi_+^{(+)} &= P''_+ a_- \xi_+ \zeta^{(+)} F(r) = (\mathcal{P}_\perp - \mathcal{G}/2)\psi_-^{(-)}, \\ Q'_+\psi_+^{(-)} &= Q'_+\psi_+^{(+)} = Q'_-\psi_-^{(+)} = Q'_-\psi_-^{(-)} = 0\,; \\ Q'_-\psi_+^{(3)} &= p_+\psi_-^{(3)}, \quad Q'_+\psi_-^{(3)} = p_-\psi_+^{(3)}. \end{aligned} \tag{63}$$

This action simultaneously changes the signs of the spin and the $(I^b\lambda^b)$ projection so that the energy of the state remains unchanged. The eigenvalues q_i of the supercharge operators follow from these actions:

$$q_1 = q'_1 = \mathcal{P}_\perp + \mathcal{G}/2\,, \quad q_2 = q'_2 = \mathcal{P}_\perp - \mathcal{G}/2\,.$$

If we recall the relationship between the degree n of the degeneracy of the spectrum and the number N of the supercharge operators $Q'_\pm$,

$$n = 2^{[N/2]},$$

where $[N/2]$ denotes the integer part of $N/2$, then in this choice of the supercharge operators it is easy to explain the twofold degeneracy of the energy spectrum, which we have discussed in the first section, as a result of the supersymmetry in the Hamiltonian.[5] Thus, the action of the supercharge operators (63) ensure the superpartnership of the energy states $\psi_+^{(-)}$ with $\psi_-^{(+)}$ and $\psi_+^{(+)}$ with $\psi_-^{(-)}$.

Using (59) we can divide the Hamiltonian H'_1 into bosonic H_B and fermionic H_F parts:

$$\begin{aligned} H'_1 &= \{Q'_+, Q'_-\} = 2\mathcal{G}\mathcal{P}_\perp b_+ b_- - 2\lambda^3\mathcal{G}\mathcal{P}_\perp a_+ a_- \\ &= 2\mathcal{G}\mathcal{P}_\perp\left(b_+ b_- + \frac{1}{2}\right) + 2\mathcal{G}\mathcal{P}_\perp\left((-\lambda^3)a_+ a_- - \frac{1}{2}\right) \\ &= H_\mathrm{B} + H_\mathrm{F}\,. \end{aligned} \tag{64}$$

The appearance of the $(-\lambda^3)$ factor in the fermionic part is connected with the different commutation rules for the bosonic operators of the $\zeta^{(-)}$ and $\zeta^{(+)}$ states. Actually, H_B also contains a term proportional to λ^3. These two Hamiltonians commute[6], $[H_\mathrm{B}, H_\mathrm{F}] = 0$, and they can be considered as Hamiltonians of two independent oscillators having the same frequency $\omega = (2\mathcal{G}\mathcal{P}_\perp)^{1/2}$:

$$\begin{aligned} H_\mathrm{B} &= \omega^2\left(b_+ b_- + \frac{1}{2}\right), & E_\mathrm{B} &= \omega^2\left(n_\mathrm{B} + \frac{1}{2}\right); \\ H_\mathrm{F} &= \omega^2\left((-\lambda^3)a_+ a_- - \frac{1}{2}\right), & E_\mathrm{F} &= \omega^2\left((-\lambda^3)n_\mathrm{F} - \frac{1}{2}\right). \end{aligned} \tag{65}$$

In respect to the supersymmetry each quantum mechanical state of the particle with definite energy is described by the bosonic and fermionic quantum numbers n_B and n_F , which accept the values [41] $n_\mathrm{B} = 0, 1, 2, 3, \ldots;\ n_\mathrm{F} = 0, 1$. As is seen from (65) and (64), this takes place for H'_1 and its eigenstates $\psi_\pm^{(\pm)}$ as well. The action of the supercharge operators change these quantum numbers as follows:

$$\begin{aligned} &Q_+(n_\mathrm{B}, n_\mathrm{F}) = (n_\mathrm{B} - 1, n_\mathrm{F} + 1)\,, \\ &Q_+(n_\mathrm{B} - 1, n_\mathrm{F} + 1) = 0\,; \\ &Q_-(n_\mathrm{B} - 1, n_\mathrm{F} + 1) = (n_\mathrm{B}, n_\mathrm{F})\,, \\ &Q_-(n_\mathrm{B}, n_\mathrm{F}) = 0\,, \end{aligned} \tag{66}$$

but the total energy of the bosonic and fermionic oscillators remains unchanged in this action. If we label the states $\psi_\pm^{(\pm)}$ by these quantum numbers as follows:

$$\psi_-^{(-)}, \psi_-^{(+)} \to (n_\mathrm{B}, n_\mathrm{F})\,; \quad \psi_+^{(-)}, \psi_+^{(+)} \to (n_\mathrm{B} - 1, n_\mathrm{F} + 1),$$

then the two action rules of the supercharge operators (66) and (63) will agree.

5 Discussion

The non-diagonal generators of the color group make the equations in Yang–Mills quantum mechanics non-diagonal as well and we decided to diagonalize the equation of motion for this case. Relying on the hermiticity of the Hamiltonian, we find the unitary transformation which diagonalizes this equation. Of course, this transformation transforms the color structure of the Hilbert space and the transformed basis vectors are the eigenfunctions of the diagonal Hamiltonian. This enabled us to establish the correspondence between the eigenvalues of the Hamiltonian and its eigenvectors.

The diagonal form of the Hamiltonian turned out to be useful for the study of supersymmetry in the case considered. It allowed us to construct the supercharge operators mapping eigenvectors corresponding to the same eigenvalues into each other. The superpartner property of the eigenvectors was easily revealed in this formulation of the supersymmetry. Also it became possible to divide the diagonal Hamiltonian into two commuting parts, corresponding to the two oscillators of bosonic and fermionic states of supersymmetry.

A number of papers [20–26] have been devoted to the study of supersymmetric Yang–Mills quantum mechanics in connection with the conjecture of Banks, Fischler, Shenker and Susskind concerning the equivalence of M-theory and $D = 10$ supersymmetric Yang–Mills quantum mechanics [49]. We hope that the study of the example of supersymmetry which we have made here will be useful for the further study of supersymmetry in Yang–Mills quantum mechanics in connection with the BFSS conjecture.

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[5] In [18, 19, 50] this degeneracy was related with the supersymmetry, but appropriate supercharge operators responsible for this degeneracy had not been found, since the energy states was not determined.

[6] H_B and H_F do not commute in another basis [50].

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References

1. G.K. Savvidy, Phys. Lett. B **71**, 133 (1979)
2. H.B. Nielsen, P. Olesen, Nucl. Phys. B **152**, 75 (1979)
3. N.K. Nielsen, P. Olesen, Nucl. Phys. B **144**, 376 (1978)
4. O. Nachtmann, in: Lectures on QCD Applications, ed. by F.L.H. Griesshammer, D. Stoll (Springer, 1997)
5. V.I. Demchik, V.V. Skalozub, Eur. Phys. J. C **25**, 291 (2002) [hep-ph/0110280]
6. V.I. Demchik, Eur. Phys. J. C **27**, 601 (2003) [hep-ph/0208274]
7. S. Mamedov, Eur. Phys. J. C **30**, 583 (2003) [hep-ph/0306179]
8. S. Mamedov, Eur. Phys. J. C **33**, 537 (2004) [hep-ph/0310273]
9. L.S. Brown, W.I. Weisberger, Nucl. Phys. B **157**, 285 (1979)
10. M. Reuter, C. Wetterich, Phys. Lett. B **334**, 412 (1994)
11. A.A. Sokolov, I.M. Ternov, V.C. Zhukovskii, Kvantovaya Mechanica (M. Nauka, 1979)
12. H.J.W. Müller-Kirsten, Introduction to Quantum Mechanics: Schrödinger Equation and Path Integral (World Scientific, 2006)
13. A. Bohm, Quantum Mechanics: Foundations and Applications (Springer, 1986)
14. S.S. Agaev, A.S. Vshivtsev, V.C. Zhukovsky, Yad. Fiz. **36**, 1023 (1982)
15. S.S. Agaev, A.S. Vshivtsev, V.C. Zhukovsky, O.F. Semyonov, Sov. Phys. J. **28**, 29 (1985)
16. S.S. Agaev, A.S. Vshivtsev, V.C. Zhukovsky, O.F. Semyonov, Sov. Phys. J. **28**, 67 (1985)
17. A.S. Vshivtsev, V.C. Zhukovskii, P.G. Midodashvili, A.V. Tatarintsev, Sov. Phys. J. **5**, 47 (1986)
18. S. Mamedov, J.-Z. Zhang, V.C. Zhukovskii, Mod. Phys. Lett. A **16**, 845 (2001) [hep/th-0101183]
19. V.C. Zhukovskii, Sov. Phys. JETP **63**, 663 (1986)
20. J. Wosiek, Nucl. Phys. B **644**, 85 (2002) [hep-th/0203116]
21. M. Campostrini, J. Wosiek, Nucl. Phys. B **703**, 454 (2004) [hep-th/0407021]
22. M. Campostrini, J. Wosiek, Phys. Lett. B **619**, 171 (2005) [hep-th/0503236]
23. M. Campostrini, J. Wosiek, Phys. Lett. B **550**, 121 (2002) [hep-th/0209140]
24. J. Wosiek, Int. J. Mod. Phys. A **20**, 4484 (2005) [hep-th/0410066]
25. J. Wosiek, hep-th/0510025
26. J. Wosiek, hep-th/0204243
27. C.-W. Kao, G.C. Nayak, W. Greiner, Phys. Rev. D **66**, 034 017 (2002) [hep-ph/0102153]
28. D.D. Dietrich, G.C. Nayak, W. Greiner, Phys. Rev. D **64**, 074 006 (2001) [hep-ph/0007139]
29. Q. Weng, C.-W. Kao, G.C. Nayak, H. Stoecker, W. Greiner, Int. J. Mod. Phys. E **10**, 483 (2001) [hep-ph/0009076]
30. R.G. Leigh, D. Minic, A. Yelnikov, Phys. Rev. Lett. **96**, 222001 (2006)
31. R.G. Leigh, D. Minic, A. Yelnikov, hep-th/0604060
32. P. Gaete, E. Guendelman, E. Spalucci, hep-th/0607151
33. P. Gaete, E. Spalucci, J. Phys. A **39**, 6021 (2006) [hep-th/0512178]
34. W.I. Weisberger, Nucl. Phys. B **161**, 61 (1979)
35. R. Parthasarathy, M. Singer, C. Viswanatham, Can. J. Phys. **61**, 1442 (1983)
36. T.N. Tudron, Phys. Rev. D **22**, 2566 (1980)
37. D. Ebert, V.V. Khudyakov, V.C. Zhukovsky, K.G. Klimenko, Phys. Rev. D **65**, 054 024 (2002) [hep-ph/0106110]
38. D. Ebert, V.C. Zhukovsky, O.V. Tarasov, Phys. Rev. D **72**, 096 007 (2005) [hep-ph/0507125]
39. V. Skalozub, M. Bordag, Nucl. Phys. B **576**, 430 (2000) [hep-ph/9905302]
40. M. Bordag, V. Skalozub, Eur. Phys. J. C **45**, 159 (2006) [hep-th/0507141]
41. L. Gendenshtein, I. Krive, Sov. Phys. Uspekhi **28**, 645 (1985)
42. E. Witten, Nucl. Phys. B **188**, 513 (1981)
43. L. Gendenshtein, Sov. J. Nucl. Phys. **41**, 166 (1985)
44. L. Gendenshtein, JETP Lett. **39**, 280 (1984)
45. F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. **251**, 267 (1995) [hep-th/9405029]
46. P. Salomonson, I.W. van Holten, Nucl. Phys. B **196**, 509 (1982)
47. A. Khare, J. Maharana, Nucl. Phys. B **244**, 409 (1984)
48. S.A. Mamedov, Candidate thesis (Moscow State University, 1991)
49. T. Banks, W. Fischler, S. Shanker, L. Susskind, Phys. Rev. D **55**, 6189 (1997) [hep-th/9610043]
50. V.C. Zhukovskii, Y.N. Belousov, Sov. Phys. J. **32**, 109 (1989)
51. R. Flume, Ann. Phys. **164**, 189 (1985)
52. Sh. Mamedov, hep-th/0611267